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TOPOLOGICAL MATRIX MODEL

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Abstract

Starting from the primal principle based on the noncommutative nature of $(9 + 1)$ -dimensional spacetime, we construct a topologically twisted version of the supersymmetric reduced model with a certain modification. Our formulation automatically provides extra $1 + 1$ dimensions, thereby the dimensions of spacetime are promoted to $10 + 2$. With a suitable gauge choice, we can reduce the model with $(10 + 2)$ -dimensional spacetime to the one with $(9 + 1)$ -dimensions and thus we regard this gauge as the light-cone gauge. It is suggested that the model so obtained would describe the light-cone F-theory. From this viewpoint we argue the relation of the reduced model to the matrix model of M-theory and the $SL(2, Z)$ symmetry of type IIB string theory. We also discuss the general covariance of the matrix model in a broken phase, and make some comments on the background independence.

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1 INTRODUCTION

The background independence is the most significant implication of the inclusion of quantum gravity in string theory. The geometry of spacetime should not be set up a priori, rather it is generated by a highly nonperturbative effect, the condensation of strings. Thus the understanding of the background independence is promisingly the key ingredient to seek the underlying principle of nonperturbative string theory.

Although the matrix model of M-theory [1] and IIB matrix models [2]–[5] may provide possible descriptions of the underlying theory of string theory, they are still lacking the fundamental principle and the understanding of the background independence. The matrix model of M-theory, however, might have provided some clues to these problems. An indication is the existence of the more fundamental degrees of freedom, D-particles or partons, from which strings are constituted. As discussed in [6], this suggests the emergence of the scales shorter than string length. More remarkably they exhibit the *noncommutative* nature of spacetime [7], and thus they are considered to be inherently non-local or fuzzy objects. This property is quite desirable to keep the fine upshots of perturbative string theory, such as the ultraviolet finiteness and T-duality [8], stemming from the extended character of strings.

These observations tempted us to look for the underlying principle based on the non-commutativity of spacetime. In the present work, we consider (anti-) D-instantons as the fundamental degrees of freedom instead of D-particles. D-particles should be constituted from D-instantons, just in the same spirit as the strings from D-particles in the matrix model of M-theory. The fuzzy instantons are represented by pure matrices, that is, matrices without continuous parameters. They are nothing other than the coordinates of a noncommutative spacetime. We shall take them as the only elements to construct our model, and impose a symmetry of arbitrary deformations of matrices. Thus our construction of the model follows that of a topological quantum field theory [9].

In section 2 we will construct a topologically twisted version of the supersymmetric reduced model with a certain modification. We argue that the critical dimensions of spacetime, including the signature, would be restricted to some extent by our formulation of the model. We make a brief remark on an additional term which is missing in the supersymmetric reduced model.

Our formulation suggests that the model we constructed would provide a possible

description of F-theory [10][11] in the light-cone gauge. In section 3 we discuss some aspects of the matrix models from the viewpoint of the F-theory interpretation of our model. Our arguments are concerning the relation of the reduced model to the matrix model of M-theory and the $SL(2, Z)$ symmetry of type IIB string theory.

In section 4 we analyze the general covariance of the matrix model in a specific class of backgrounds. We show the physical equivalence of the backgrounds in this class connected by the general coordinate transformations. The topological symmetry plays a central role in this analysis. Although our analysis is limited to the backgrounds of commuting matrices, this supports the expectation that our model as a whole is independent of the backgrounds by virtue of the topological symmetry.

While carrying out the present work, we became aware of the work [12], in which they also constructed a topological matrix model starting from the fundamental principle in a spirit quite close to ours.

2 TOPOLOGICAL MODEL

Topological model is the most promising candidate for the background independent description of the underlying theory of strings [13][14]. As stated in the introduction, the constituents of our model are fuzzy instantons represented by the coordinates of a non-commutative spacetime, $N \times N$ hermitian matrices X^μ , where the index μ runs from 0 to 9.¹ We shall hypothesize that the underlying symmetry is a topological symmetry, that is, arbitrary deformations of the noncommutative coordinates X^μ :

$$\delta X^\mu = \epsilon^\mu. \tag{1}$$

where ϵ^μ 's are arbitrary $N \times N$ hermitian matrices.

In what follows we will construct a topologically twisted version of supersymmetric reduced model with a certain modification. We shall take the action which has the large symmetry (1) to be identically zero,

$$S = 0, \tag{2}$$

and carry out the BRST gauge fixing of the symmetry (1) with this action, according to [15] and [16]. Now let us introduce the BRST transformation laws,

$$\delta X^\mu = \psi^\mu, \quad \delta \psi^\mu = 0, \tag{3}$$

¹The dimension and the signature of spacetime are constrained to some extent, as we will discuss below.

where the fields X^μ and ψ^μ are $N \times N$ hermitian matrices and the ghost numbers of them are 0 and 1, respectively. Since the gauge fixed action associated with this BRST symmetry will have residual gauge symmetries, we will perform a second stage of gauge fixing later.²

To fix the topological symmetry, it seems natural to choose certain “self-dual” equations as the gauge conditions. The “self-dual” equation is a variant of the higher dimensional analogue of the self-dual equation in four dimensions given in [17]:

$$[X^\mu, X^\nu] = \frac{1}{2} T^{\mu\nu\rho\sigma} [X_\rho, X_\sigma]. \quad (4)$$

Here we define the totally antisymmetric tensor $T^{\mu\nu\rho\sigma}$ as

$$T^{\mu\nu\rho\sigma} = (\zeta^T, 0) \Gamma^{\mu\nu\rho\sigma} \begin{pmatrix} \zeta \\ 0 \end{pmatrix}, \quad (5)$$

where $\Gamma^{\mu\nu\rho\sigma}$ is the totally antisymmetric product of Γ matrices for $SO(9, 1)$ spinor representation, and ζ is a unit constant Majorana-Weyl spinor, $\zeta^T \zeta = 1$. We will further impose that ζ satisfies a $SO(8)$ Weyl condition.

Let us decompose the gamma matrices Γ^μ , in terms of the $SO(8)$ ones γ^i , into $\Gamma^0 = i\sigma_2 \otimes \mathbf{1}_{16}$, $\Gamma^i = \sigma_1 \otimes \gamma^i$, and $\Gamma^9 = \sigma_1 \otimes \gamma^9$. Then one can easily find that the 4-th rank antisymmetric tensor (5) is broken up into

$$T^{0ijk} = T^{0ij9} = T^{ijk9} = 0, \quad T^{ijkl} = \zeta^T \gamma^{ijkl} \zeta, \quad (6)$$

where $i, j, k, l = 1, \dots, 8$ and the tensor T^{ijkl} is invariant under $SO(7)$ rotation as is obvious from its definition. The appearance of $SO(7)$ is understood from the construction of the “self-dual” equation in eight dimensions discussed in detail in [17].

As the result we obtain the following explicit expression for our gauge conditions (4):

$$F_{09} = F_{0i} = F_{9i} = 0, \quad (7)$$

$$\left\{ \begin{array}{l} F_{12} + F_{34} + F_{56} + F_{78} = 0, \\ F_{13} + F_{42} + F_{57} + F_{86} = 0, \\ F_{14} + F_{23} + F_{76} + F_{85} = 0, \\ F_{15} + F_{62} + F_{73} + F_{48} = 0, \\ F_{16} + F_{25} + F_{38} + F_{47} = 0, \\ F_{17} + F_{82} + F_{35} + F_{64} = 0, \\ F_{18} + F_{27} + F_{63} + F_{54} = 0, \end{array} \right. \quad (8)$$

²Strictly speaking, they are not gauge symmetries, because the base manifold of our model is a point and there are no local symmetries.

where we define the field strengths $F_{\mu\nu} = i[X_\mu, X_\nu]$.

Some remarks are in order: (i) The latter set of the gauge conditions (8) is used in the context of the cohomological Yang-Mills theory in eight dimensions [18], where they constructed the nearly topological Yang-Mills theory, in particular, on the Joyce manifold with $spin(7)$ holonomy.³ (ii) The second rank tensor F_{ij} in eight dimensions belongs to **28** of $SO(8)$, whose $SO(7)$ decomposition is **21** \oplus **7**. A set of seven equations (8) belongs to the **7**, and remarkably it enjoys the octonionic structure as noted in [17]. The appearance of the octonion may explain that the dimensions $9 + 1$ of spacetime is critical, including the signature, along the line of the argument given in [20].⁴ (iii) Since the base manifold of our model is a point, the “self-dual” equation does not necessarily mean the instanton equation in the field theory sense, rather it is only formal analogue of that in higher dimensions. We note, however, that there are possibilities the “self-dual” equation does indeed become the instanton equation in the field theory sense, if we take certain large N limits which give, say, the configurations of matrices X ’s corresponding to the toroidal compactifications of the matrix model of M-theory.[1][21]

In order to construct a gauge fixed action for the topological symmetry, we must introduce the antighosts $\chi_{\mu\nu}$ with the ghost number -1 and the Nakanishi-Lautrup fields $b_{\mu\nu}$, whose BRST transformation rules are

$$\delta\chi_{\mu\nu} = ib_{\mu\nu}, \quad \delta b_{\mu\nu} = 0. \quad (9)$$

They satisfy the “anti self-dual” equations:

$$\chi_{ij} = -\frac{1}{6}T_{ijkl}\chi^{kl}, \quad b_{ij} = -\frac{1}{6}T_{ijkl}b^{kl}, \quad (10)$$

or equivalently,

$$\frac{1}{4}\left(\delta_{ik}\delta_{jl} - \frac{1}{2}T_{ijkl}\right)\chi^{kl} = \chi_{ij}, \quad (11)$$

$$\frac{1}{4}\left(\delta_{ik}\delta_{jl} - \frac{1}{2}T_{ijkl}\right)b^{kl} = b_{ij}. \quad (12)$$

The operator $P_{ijkl} = \frac{1}{4}\left(\delta_{ik}\delta_{jl} - \frac{1}{2}T_{ijkl}\right)$ is a projection operator onto the subspace of the eigenvalue -3 of T_{ijkl} .⁵

³See also [19] for related discussions for the “self-dual” equations.

⁴As advertised in the abstract, the extra $1 + 1$ dimensions arise automatically in our formulation, in addition to $9+1$ dimensions. Thus this may in turn account for that the critical dimension is $10 + 2$.

⁵The projection operator for the eigenvalue 1 is $\frac{3}{4}\left(\delta_{ik}\delta_{jl} + \frac{1}{6}T_{ijkl}\right)$. In order to write the projection operators in $SO(9, 1)$ covariant way, we need to make them of a quadratic form in $T^{\mu\nu\rho\sigma}$, because there are three distinct eigenvalues of $T^{\mu\nu\rho\sigma}$.

Now the gauge fixed action is

$$S_{GF} = -i\delta\text{Tr}\left\{\frac{1}{4}\chi_{\mu\nu}\left(F^{\mu\nu} - \frac{1}{2}T^{\mu\nu\rho\sigma}F_{\rho\sigma}\right) + \frac{1}{2}\alpha_{\mu\nu}\chi_{\mu\nu}b^{\mu\nu}\right\}, \quad (13)$$

where $\alpha_{\mu\nu}$'s are gauge fixing parameters and they are not components of a second rank tensor. The normalization of the action is in conformity with that of the projection operator P_{ijkl} .

For later purpose we take the Landau gauge for the gauge fixing functions of (7), $\alpha_{09} = \alpha_{0i} = \alpha_{9i} = 0$, and set the gauge parameters $\alpha_{ij} = \alpha$ for those of (8). Then integrating out the auxiliary fields b_{09} , b_{0i} , and b_{9i} , the gauge fixed action (13) reduces to

$$S_{GF} = \text{Tr}\left\{\frac{1}{4}b_{ij}\left(F^{ij} - \frac{1}{2}T^{ijkl}F_{kl}\right) + \frac{\alpha}{2}b_{ij}b^{ij} - \frac{1}{4}\chi_{ij}\left([X^{[i}, \psi^{j]}] - \frac{1}{2}T^{ijkl}[X_{[k}, \psi_{l]}]\right) - 2\chi_{09}[X^{[0}, \psi^{9]}] - 2\chi_{0i}[X^{[0}, \psi^{i]}] - 2\chi_{9i}[X^{[9}, \psi^{i]}]\right\} \quad (14)$$

$$= \text{Tr}\left\{b_{ij}F^{ij} + \frac{\alpha}{2}b_{ij}P^{ijkl}b_{kl} - \chi_{ij}[X^{[i}, \psi^{j]}] - 2\chi_{09}[X^{[0}, \psi^{9]}] - 2\chi_{0i}[X^{[0}, \psi^{i]}] - 2\chi_{9i}[X^{[9}, \psi^{i]}]\right\}, \quad (15)$$

with the constraints $\delta([X^0, X^9]) \times \delta([X^0, X^i]) \times \delta([X^9, X^i])$.

Now let us look on the delta function constraints,

$$[X^0, X^9] = 0, \quad [X^0, X^i] = 0, \quad [X^9, X^i] = 0. \quad (16)$$

Using a $U(N)$ gauge rotation, we can choose a basis in which X^0 and X^9 take their values on the Cartan subalgebra of $U(N)$. In such a basis the second and third equations in (16) constrain the values of X^i 's on the Lie algebra of $U(N-M) \otimes U(1)^M$, ($M = 0, \dots, N$), depending on the values of X^0 and X^9 . For later convenience, we will denote the gauge groups $U(N-M) \otimes U(1)^M$, ($M = 0, \dots, N$) as $\mathcal{U}(N)$ collectively. Having this in mind and taking into account the fact that the integrations over the fermions in the second line of the gauge fixed action (15) give the Jacobian factors for the delta functions, we can further reduce the gauge fixed action to

$$S_{GF} = \text{Tr}\left\{b_{ij}F^{ij} + \frac{\alpha}{2}b_{ij}P^{ijkl}b_{kl} - \chi_{ij}[X^{[i}, \psi^{j]}]\right\}, \quad (17)$$

where all fields take their values on the Lie algebra of $\mathcal{U}(N)$.

Note that X^0 and X^9 do not completely disappear from the system. The diagonal part of them do remain, and in some cases left their traces of $U(1)$ factors in the gauge

symmetry. This is reminiscent of the light-cone gauge of string theory, in which only do the zero modes of the light-cone coordinates survive. Indeed it will be suggested that we can regard this gauge as the light-cone gauge of F-theory, as we will discuss in section 3.

As mentioned previously, the gauge fixed action (17) for the topological symmetry has a fermionic gauge symmetry, $\delta_\lambda \psi^i = [X^i, \lambda]$, $\delta_\lambda b_{ij} = \{\chi_{ij}, \lambda\}$, and $\delta_\lambda X^i = \delta_\lambda \chi_{ij} = 0$, where λ is a Grassmann-valued matrix. In fact the variation of the action under this transformation is BRST-exact. In order to fix this symmetry, we must introduce a ghost ϕ for ghost ψ^i , whose BRST transformation laws are

$$\begin{aligned}\tilde{\delta} X^i &= 0, & \tilde{\delta} \psi^i &= [X^i, \phi], & \tilde{\delta} \phi &= 0, \\ \tilde{\delta} \chi^{ij} &= 0, & \tilde{\delta} b_{ij} &= -i[\chi_{ij}, \phi] & (\delta \phi = 0).\end{aligned}\tag{18}$$

We will choose the following gauge function for this symmetry,

$$[X_i, \psi^i],\tag{19}$$

and introduce an antighost $\bar{\phi}$ and an auxiliary field η with the BRST transformation rules,

$$\tilde{\delta} \bar{\phi} = 2\eta, \quad \tilde{\delta} \eta = \frac{1}{2}[\bar{\phi}, \phi], \quad (\delta \bar{\phi} = \delta \eta = 0).\tag{20}$$

Then the total gauge fixed action is

$$\begin{aligned}\tilde{S}_{GF} &= S_{GF} - (\delta + \tilde{\delta}) \text{Tr} \left(\frac{1}{2} \bar{\phi} [X_i, \psi^i] - \frac{1}{4} \bar{\phi} [\phi, \eta] - \frac{i}{4} b_{ij} \chi^{ij} \right) \\ &= \text{Tr} \left\{ b_{ij} F^{ij} + \frac{\alpha}{2} b_{ij} P^{ijkl} b_{kl} - \chi_{ij} [X^i, \psi^j] \right. \\ &\quad \left. - \eta [X_i, \psi^i] - \frac{1}{2} \bar{\phi} \{\psi_i, \psi^i\} - \frac{1}{2} \phi \{\eta, \eta\} - \frac{1}{4} \phi \{\chi_{ij}, \chi^{ij}\} \right. \\ &\quad \left. + \frac{1}{2} [X_i, \phi] [X^i, \bar{\phi}] + \frac{1}{8} [\phi, \bar{\phi}]^2 - \frac{1}{4} b_{ij} P^{ijkl} b_{kl} \right\},\end{aligned}\tag{21}$$

where the total BRST operator $\delta + \tilde{\delta}$ is nilpotent up to a $\mathcal{U}(N)$ gauge transformation. We have added suitable BRST exact terms in order to make the action of the standard form.

At this stage we set the gauge parameters α to be zero in such a way that the coefficient of the term $b_{ij} P^{ijkl} b_{kl}$ becomes $-1/4$. Thus our choice of gauge for the gauge conditions (8) is the Feynmann gauge. Then integrating out the auxiliary fields b_{ij} , the total gauge fixed action reduces to

$$\begin{aligned}\tilde{S}_{GF} &= \text{Tr} \left\{ \frac{1}{4} F_{ij} F^{ij} - \frac{1}{8} T^{ijkl} F_{ij} F_{kl} - \chi_{ij} [X^i, \psi^j] \right. \\ &\quad \left. - \eta [X_i, \psi^i] - \frac{1}{2} \bar{\phi} \{\psi_i, \psi^i\} - \frac{1}{2} \phi \{\eta, \eta\} - \frac{1}{4} \phi \{\chi_{ij}, \chi^{ij}\} \right. \\ &\quad \left. + \frac{1}{2} [X_i, \phi] [X^i, \bar{\phi}] + \frac{1}{8} [\phi, \bar{\phi}]^2 \right\}.\end{aligned}\tag{22}$$

This gauge fixed action still has the ordinary gauge symmetry. The gauge fixing procedure is quite standard and we will not carry out it here. We would, however, like to note that we can make the total BRST operator nilpotent off shell by adding the BRST operator for the ordinary gauge symmetry.

The Relation to The Supersymmetric Reduced Model

Now let us discuss the relation of our model to the supersymmetric reduced model. The fields in the latter are bosons which transform as a $(\mathbf{9}, \mathbf{1})$ vector under a global $SO(9, 1)$ rotation, and fermions as a $\mathbf{16}$ spinor. Under a subgroup $SO(1, 1) \otimes SO(7)$ of $SO(9, 1)$, they are decomposed into

$$(\mathbf{9}, \mathbf{1}) \longrightarrow \mathbf{8}_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_{-2}, \quad (23)$$

$$\mathbf{16} \longrightarrow \mathbf{8}_1 \oplus \mathbf{7}_{-1} \oplus \mathbf{1}_{-1}, \quad (24)$$

where the subscripts denote twice $SO(1, 1)$ spin, and we embedded $SO(7)$ into one of the spinor representations of $SO(8)$, say, $\mathbf{8}_s$.

On the other hand the fields in the former are the following:

$$X^i (\mathbf{8}_0), \quad \phi (\mathbf{1}_2), \quad \bar{\phi} (\mathbf{1}_{-2}), \quad (25)$$

$$\psi^i (\mathbf{8}_1), \quad \chi_{ij} (\mathbf{7}_{-1}), \quad \eta (\mathbf{1}_{-1}), \quad (26)$$

where the subscripts denote the ghost number.

Thus we find that the field contents of our model completely match with those of the reduced model. This shows that we can identify our model as a topologically twisted version of the supersymmetric reduced model. Indeed one can see that the action (22) is equivalent to that of the supersymmetric reduced model

$$S_{RM} = \text{Tr} \left\{ -\frac{1}{4} [A_\mu, A_\nu]^2 - \frac{1}{2} \bar{\Psi} \Gamma^\mu [A_\mu, \Psi] \right\}, \quad (27)$$

up to the term $\text{Tr} T^{ijkl} F_{ij} F_{kl}$, by the following identifications of the fields:

$$\begin{aligned} A^i &= X^i, & A_0 + A_9 &= \phi, & A_0 - A_9 &= \bar{\phi} \\ \lambda_+^i &= \psi^i, & \lambda_-^a &= 2\chi^{8a}, & \lambda_-^8 &= \eta, \end{aligned} \quad (28)$$

where $a = 1, \dots, 7$ and the $SO(8)$ chiral spinors λ_+ and λ_- are given by

$$\Psi^T = (\lambda_+^T, \lambda_-^T, \mathbf{0}, \mathbf{0}), \quad (29)$$

in the convention of the gamma matrices Γ^μ employed before.⁶

Under the identification (28) of the fields, the resulting action is written as

$$\begin{aligned}
S_{RM} = & \text{Tr} \left\{ \frac{1}{4} F_{ij} F^{ij} - \chi_{8a} \left(2[X^{[8}, \psi^{a]} + c^{abc} [X_{[b}, \psi_{c]}] \right) - \eta [X_i, \psi^i] \right. \\
& - \frac{1}{2} \bar{\phi} \{ \psi_i, \psi^i \} - \frac{1}{2} \phi \{ \eta, \eta \} - 2\phi \{ \chi_{8a}, \chi^{8a} \} \\
& \left. + \frac{1}{2} [X_i, \phi] [X^i, \bar{\phi}] + \frac{1}{8} [\phi, \bar{\phi}]^2 \right\}.
\end{aligned} \tag{30}$$

Here we have used specific representations of the gamma matrices γ^i and of the auxiliary fields χ_{ij} in terms of the structure constants c_{abc} for octonions. They are summarized in the appendix.

Note that the signature of $SO(1, 1)$ is relevant since the generator of $SO(1, 1)$ is nothing other than the ghost number current and thus must correspond to a scale, not a phase, transformation. This shows the signature of $(\phi + \bar{\phi}, \phi - \bar{\phi})$ must be $(1, 1)$.

We would also like to remark that the total gauge fixed action (22) contains an extra term $\text{Tr} T^{ijkl} F_{ij} F_{kl}$, compared with the supersymmetric reduced model. For finite N this term vanishes by virtue of the cyclicity of the trace and the Jacobi identity. It can, however, survive in certain large N limits. For instance the configurations of matrices corresponding to the toroidal compactifications break the cyclicity of the trace. Moreover, in this case, the Jacobi identity can be lifted to the Bianchi identity, and it is well-known that the Bianchi identity does not hold in the presence of the topological defects. Thus our model is endowed with an interesting modification of the supersymmetric reduced model in the large N limits.

One may, however, suspect that this extra term would violate the successful outcomes of the matrix models, such as the emergence of various brane solutions and the precise agreements of the brane-brane scatterings in the matrix models with those in supergravities. Fortunately this does not seem to happen. This is because the variations of the extra term with respect to X 's are vanishing as far as the Bianchi identity holds. Therefore, in the case of simple brane configurations such as $[P, Q] = \text{const.}$ and its generalizations, the equations of motion of the matrix models are not altered, and the extra term is independent of the detailed form of the matrices. Thus it appears that this term would not affect the successful results for the known brane solutions and the amplitudes of the brane-brane scatterings.

⁶ $\Gamma^0 = i\sigma_2 \otimes \mathbf{1}_{16}$, $\Gamma^i = \sigma_1 \otimes \gamma^i$, and $\Gamma^9 = \sigma_1 \otimes \gamma^9$.

We would, however, like to emphasize that there could be some effects of the extra term $\text{Tr} T^{ijkl} F_{ij} F_{kl}$ on the dynamics of the matrix model, if the configurations of matrices break the Bianchi identity. In this case even the equations of motion are modified and we expect that the extra term would produce brane solutions yet unknown or missing.

3 F-THEORY INTERPRETATION AND IIB STRING

In addition to the $(9 + 1)$ -dimensional coordinates X^μ , there emerge two extra bosons ϕ and $\bar{\phi}$ with the signature $(1, 1)$. This is likely to indicate that the spacetime dimensions of our model are promoted to $10 + 2$. Thus we are tempted to interpret our model as a possible description of F-theory [10][11]. As we took the light-cone gauge in our formulation, we suggest that our model would describe the light-cone F-theory with $9 + 1$ transverse dimensions. This viewpoint may illuminate the understanding of some aspects of the matrix models.

1. The Relation of The Reduced Model to The Matrix Model of M-Theory

So far there are no arguments to directly connect F-theory with M-theory without compactifying M-theory. Our viewpoint, however, may provide a way to relate them directly. One naively expect that the compactification of one time direction of F-theory leads to M-theory in $10 + 1$ dimensions. Indeed our viewpoint supports this idea as follows:

Let us look on the bosons in our model. We will list them below.

$$\begin{array}{cc|cc} (X^0, X^9) & & X^i & (\phi + \bar{\phi}, \phi - \bar{\phi}) \\ 1 + 1 & & 8 & 1 + 1 \\ \text{light-cone} & & \text{transverse} & \text{transverse} \end{array} \quad (31)$$

The reduced model is lifted to a $(1 + 0)$ -dimensional supersymmetric Yang-Mills theory (SYM_{1+0}) by the compactification of one time direction $\phi + \bar{\phi}$, in which the matrix $\phi + \bar{\phi}$ is represented by a covariant derivative $-i\partial_t - A_0(t)$ [1][21]. Since SYM_{1+0} is nothing other than the matrix model of M-theory with $9 + 0$ transverse directions, this shows the relation of M-theory to F-theory compactified on a timelike S^1 :

$$\text{LC F} \xrightarrow{S^1} \text{LC M}, \quad (32)$$

where LC denotes the light-cone gauge and S^1 is in a timelike direction.

The connection of the reduced model with the matrix model of M-theory was also anticipated in [2].

2. $SL(2, Z)$ Symmetry of Type IIB String

Type IIB string theory would be obtained by the compactification of F-theory on a $(1, 1)$ space [10].⁷ In the light of our F-theory interpretation, a matrix description of the light-cone type IIB theory is expected to be the reduced model on $T^{1,1}$ torus, where ϕ - and $\bar{\phi}$ -directions are compactified. Thus the light-cone type IIB theory seems to be described by a $(1 + 1)$ -dimensional $N = 8$ supersymmetric Yang-Mills theory (SYM_{1+1}) discussed as the type IIA string in [23]–[25]. In the context of the matrix model of M-theory, the authors in [26][25] discussed that the type IIB string would be reproduced by a $(2 + 1)$ -dimensional supersymmetric Yang-Mills theory on a 2-torus, in which one cycle of the torus was taken much smaller than the other. Although our viewpoint is different from theirs, we expect that both of them would be linked to each other.

To precisely interpret SYM_{1+1} as the light-cone type IIB string (both D- and F-), we have to explain how the correct chirality comes about in a $T^{1,1}$ compactification. Here we only assume that the correct chirality could be obtained in our framework.

Now let us discuss the $SL(2, Z)$ symmetry of type IIB strings. As conjectured in [10], the $SL(2, Z)$ symmetry is expected to be understood as a geometrical symmetry of the torus $T^{1,1}$. In the case at hand we might be able to support this conjecture in the following way:

Let the radius of $(\phi + \bar{\phi}, \phi - \bar{\phi})$ be (R_+, R_-) . The moduli parameter of the torus $T^{1,1}$ is given by $\tau = iR_+/R_-$. The compactification of the reduced model on S^1 with the radius R_+ leads to the matrix model of M-theory as mentioned above. Subsequently we compactify the matrix model of M-theory on S^1 with the radius R_- . Then the radius R_- is related to the string coupling constant g_s via $R_- = g_s$ in string unit. Thus a modular transformation $\tau \rightarrow -1/\tau$ gives the S-duality $g_s \rightarrow 1/g_s$ as we expected.

A remark is in order. Type IIB superstring field theory was derived from the IIB matrix model in [27].⁸ They performed the light-cone decomposition of the coordinates in order to connect the IIB matrix model with the light-cone string field theory. They, however, did not take the light-cone gauge, rather they made only the formal light-cone decomposition of the coordinates. The degrees of freedom of the light-cone directions were not subtracted at all. This suggests that the reduced model with $(9 + 1)$ -dimensional

⁷The authors in [22] made a quite different proposal in this respect.

⁸We would like to thank Asato Tsuchiya for explaining their work.

spacetime contains only the degrees of freedom of the light-cone formulation.⁹ Thus our F-theory interpretation is not led to an immediate contradiction to the result obtained in [27].

4 BROKEN PHASE AND GENERAL COVARIANCE

In this section we would like to argue how we should understand the general covariance of the matrix model in our framework. Our analysis here is limited to that in a broken phase, that is, in a specific background configuration of the matrices. The understanding of the broken phase itself is important to connect our model with the physics in the real world. We, however, consider it to be complementary to the understanding of the background independence of the matrix model as well.

Now let us take a background in which the matrices are mutually commuting:

$$[x^i, x^j] = [x^i, \varphi] = [x^i, \bar{\varphi}] = [\varphi, \bar{\varphi}] = 0, \quad (33)$$

where x^i , φ , and $\bar{\varphi}$ are the background fields for X^i , ϕ , and $\bar{\phi}$ respectively.

This background is considered as a commutative spacetime limit of a noncommutative one. Thus we should be able to see the general covariance of the ordinary spacetime.

As discussed in section 2, the symmetries of our model are a topological symmetry \mathcal{T} and the gauge symmetries \mathcal{G} . The background we took breaks the symmetries down to a gauge symmetry \mathcal{H} , which commutes with the background. Under the transformations

$$\delta X^i = f^i(x^i, \varphi, \bar{\varphi}), \quad \delta \bar{\phi} = f_{\bar{\varphi}}(x^i, \varphi, \bar{\varphi}), \quad (34)$$

for the fluctuations from the background, where f^i and $f_{\bar{\varphi}}$ are arbitrary functions of the background matrices x^i , φ , and $\bar{\varphi}$, the gauge symmetry \mathcal{H} is preserved. Note that a ghost of ghost ϕ is invariant under the topological transformation. Now these transformations induce replacements of the background

$$x^i \rightarrow x^i + f^i(x^i, \varphi, \bar{\varphi}), \quad \bar{\varphi} \rightarrow \bar{\varphi} + f_{\bar{\varphi}}(x^i, \varphi, \bar{\varphi}), \quad (35)$$

which are nothing other than the general coordinate transformations. The above transformations (34) are caused by the generators of a broken symmetry, the topological symmetry \mathcal{T} . This means that the backgrounds connected with the transformations (35) are

⁹This point was also noted in [28] in the light of supersymmetry transformation of the Wilson loops.

equivalent to each other up to the BRST transformations. Thus they all are physically equivalent. This shows the general covariance of the matrix model in a broken phase. The key ingredient here is the topological symmetry \mathcal{T} .

We should not, however, expect the full general covariance of the $(10+2)$ -dimensional spacetime because of the invariance of ϕ under the topological transformation. We also remark that our derivation has been performed in the light-cone gauge. However the inclusion of the light-cone coordinates seems to have no obstruction for the above arguments. We do not now have a definite answer to a question whether the general covariance we discussed is simply $(9+1)$ - or intricately $(10+1)$ -dimensional one. We would also like to mention that the analysis in this section indicates the topological symmetry \mathcal{T} is a key to understand the background independence of the matrix model, as it should be.

5 CONCLUSIONS AND DISCUSSIONS

We proposed a candidate for the primal principle to define the underlying theory of string theory. Our first hypothesis is that the most fundamental constituents of the theory are fuzzy instantons represented by the matrices which denote a position in a noncommutative spacetime. The second one is that the theory has a topological symmetry, that is, arbitrary deformations of the fields, which is likely to be the maximum one considered naturally. The resulting theory is a topologically twisted version of the supersymmetric reduced model with a certain modification. There emerged extra $1+1$ spacetime dimensions in addition to the starting $9+1$ dimensions, thereby we were tempted to interpret our model as a matrix description of F-theory.

Until now F-theory is defined only through the compactifications on elliptically fibered complex manifolds. To verify our interpretation, we must study the compactifications of our model on elliptically fibered surfaces. In this connection, it would be interesting to study the compactification of our model on a K_3 orbifold, T^4/Z_2 [29].

Although we took the light-cone gauge in our formulation, it does not seem to be difficult to construct our model in covariant way. This appears to be easily realized by taking the Feynmann gauge for all the gauge conditions, instead of taking the Landau gauge for those containing the light-cone coordinates. This is only a choice of gauge, and thus two formulations would be physically equivalent. We expect that a covariant formulation of our model sheds some lights on that of the matrix model of M-theory.

Far from being a problem, there is another formulation to obtain a topologically twisted supersymmetric reduced model. Starting with only transverse 8-dimensional space, we can construct almost the same model as the one in the light-cone gauge. The difference arises in the gauge group. This alternative formulation gives the model with $U(N)$ gauge symmetry, not $\mathcal{U}(N)$. The formulation itself is mathematically more beautiful than that performed in the present paper. In this formulation, the $(9+1)$ -dimensional whole world is emerged as a hologram of the 8-dimensional transverse space. This gives a concrete realization of the world as a hologram [30][31]. It is amusing but quite different from the one in the present work to interpret our model in this manner.

We would like to mention the physical states of our model. They are defined as the states that are invariant under the BRST transformation up to a gauge transformation. Since the BRST charge is a supercharge which is singlet under $SO(7)$ rotation, the physical states are all BPS states that preserves at least $1/16$ of the supersymmetry. There are no non-BPS states in our model. Considering the connection to the real world, the BRST symmetry must be broken spontaneously. By this mechanism four dimensional spacetime should be dynamically generated. This problem extremely deserves to be investigated further.

We discussed the general covariance of the matrix model. Although our analysis was limited to that in a broken phase, we want to study the general covariance in the unbroken phase. The topological symmetry must be a key to show the background independence of the matrix model. Now we believe that the background independence has been already encoded in our formulation of the matrix model.

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APPENDIX

In this appendix, we will make a brief summary of some definitions and formulae concerning the octonion used in section 2.

The octonion basis, e_a ($a = 1, \dots, 7$) and $e_8 = 1$, satisfy

$$e_a e_b = -\delta_{ab} + c_{abc} e_c, \quad (\text{A.1})$$

where c_{abc} are the structure constants for octonions and totally antisymmetric.

Next we will define the following 8×8 matrices:

$$\begin{aligned} (t^a)_{bc} &= c_{abc}, \\ (t^a)_{b8} &= -(t^a)_{8b} = \delta_{ab}, \\ (t^a)_{88} &= 0, \end{aligned} \quad (\text{A.2})$$

The $SO(8)$ gamma matrices γ^i are expressed, in terms of these 8×8 matrices, as

$$\gamma^a = \begin{pmatrix} \mathbf{0} & t^a \\ -t^a & \mathbf{0} \end{pmatrix}, \quad \gamma^8 = \begin{pmatrix} \mathbf{0} & \mathbf{1}_8 \\ \mathbf{1}_8 & \mathbf{0} \end{pmatrix}. \quad (\text{A.3})$$

Then one can find that the 4-th rank antisymmetric tensors T^{ijkl} defined in (6) enjoy an expression in terms of the structure constants c_{abc} , when picking a Majorana-Weyl spinor $\zeta_\alpha = \delta_{8\alpha}$:

$$\begin{aligned} T^{8abc} &= c_{abc} \\ T^{abcd} &= \frac{1}{3} (-c_{abe} c_{ecd} + c_{ace} c_{ebd} - c_{ade} c_{ebc}). \end{aligned} \quad (\text{A.4})$$

Note that in this representation of T^{ijkl} the gauge conditions (8) are given succinctly by

$$F_{8a} = \frac{1}{2} c_{abc} F^{bc}, \quad (\text{A.5})$$

and the auxiliary fields χ_{ij} satisfy

$$\chi_{ab} = c_{abc} \chi^{c8}. \quad (\text{A.6})$$

Lastly we will list two useful identities below:

$$\begin{aligned} c_{acd} c_{bcd} &= 6\delta_{ab} \quad \Leftrightarrow \quad \text{tr}(\{t^a, t^b\}) = -16\delta_{ab}, \\ c_{adf} c_{bfe} c_{ced} &= -3c_{abc} \quad \Leftrightarrow \quad \text{tr}(t^a t^b t^c) = 0. \end{aligned} \quad (\text{A.7})$$

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